Pick 3 or Pick 2 ... another NP–complete 3-Dimensional Matching Variant

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Abstract

We prove that the 3-DIMENSIONAL MATCHING problem remains NP–complete if the matching can be built not only with triples $(x, y, z) \in M \subseteq \{X \times Y \times Z\}$, but also choosing one of their two corresponding pairs (x, y) or (x, z). We also discuss some other variants of the problem.

1 Problem definition

The following variant of the 3-DIMENSIONAL MATCHING problem (3DM) was posted on cstheory.stackexchange.com, a question and answer site for professional researchers in theoretical computer science and related fields:

Definition 1.1 (3-DIMENSIONAL MATCHING VARIANT, 3DMV)**.**

Input: Set $M \subseteq X \times Y \times Z$ where X, Y, Z are disjoint sets; we call $M_{XY} = \{(x, y) \mid \exists z \text{ s.t. } (x, y, z) \in M\}$ the set of pairs of $X \times Y$ that appear in the triples of M, and $M_{XZ} = \{(x, z) \mid \exists y \text{ s.t. } (x, y, z) \in M\}$ the set of pairs of $X \times Z$ that appear in the triples of M.

Question: Does there exist a set $M' \subseteq M \cup M_{XY} \cup M_{XZ}$ such that every element of $X \cup Y \cup Z$ is included in a triple or a pair of M' exactly once?

Informally we want to build an exact cover of $X \cup Y \cup Z$ using the triples of M or one of the two pairs $(x, y), (x, z)$ that are contained in a triple $(x, y, z) \in M$.

In the next section we prove that 3DMV is NP–complete.

2 NP-completeness proof

First it is easy to see that the problem 3DMV is in NP: given a solution we can check that it is valid in polynomial time. We prove that it is NP–hard giving a reduction from the NP–complete problem SET COVER [1]:

Definition 2.1 (SET COVER, SC)**.**

Input: An universe A of n elements: $A = \{a_1, ..., a_n\}$, a collection of m subsets $\mathcal{S} = \{S_1, S_2, ..., S_m\}$ (with $S_i \subseteq A$) and an integer k.

Question: Does there exist a sub-collection $\mathcal{C} \subseteq \mathcal{S}$ of size at most k (i.e. $|\mathcal{C}| \leq k$) such that $\bigcup_{S_i \in \mathcal{C}} S_i = A$?

Given an instance of SC we will build in polynomial time an instance of 3DMV that has a solution if and only if the original instance of SC has a solution. We will use the superscript x (resp. y, z) to denote an element of the set X (resp. Y, Z). We will use dum^y (resp. dum^z) to denote an *unique* new element that is included in the set Y (resp. Z).

We start adding n *universe elements* $a^z_1, a^z_2, ..., a^z_n$ to Z that correspond to the elements of the set A. Then for each set $S_i = \{a_{i_1}, a_{i_2}, ..., a_{i_q}\}$ we build the following *set selector gadget*:

- add q *set elements* $S^x_{i_1}, S^x_{i_2}, ..., S^x_{i_q}$ to X, one *selector element* R^y_i and q *link elements* $L^y_{i_1}, L^y_{i_2}, ..., L^y_{i_q}$ to set Y;
- add the triples:

$$\begin{array}{llll} t_{i_1}: & (S^x_{i_1}, R^y_i, dum^z) & t'_{i_1}: & (S^x_{i_1}, L^y_{i_1}, a^z_{i_1}) \\ t_{i_3}: & (S^x_{i_2}, L^y_{i_1}, dum^z) & t'_{i_2}: & (S^x_{i_2}, L^y_{i_2}, a^z_{i_2}) \\ & \ldots & & \ldots \\ t_{i_q}: & (S^x_{i_q}, L^y_{i_{q-1}}, dum^z) & t'_{i_q}: & (S^x_{i_q}, dum^y, a^z_{i_q}) \end{array}$$

In order to include all the set elements S^x_i exactly once, one of the two triples t_{i_j}, t'_{i_j} must be in M' (or one of the corresponding pairs), but not both. But if we include the first triple $(S^x_{i_1}, R^y_i, dum^z)$ (or one of the corresponding pairs $(S^x_{i_1}, dum^z)$, $(S^x_{i_1}, R^y_i)$) the only way to include $L^y_{i_1}$ is to include the triple t_{i_3}; but if we include t_{i_3} then the only way to include $L^y_{i_2}$ is to include the triple t_{i_4} and so on. In other words, if we "choose" t_{i_1} then we cannot include any of the $a^z_{i_j}$ included in the triple t'_{i_j}.

On the contrary, if we pick t'_{i_1}, then we can also pick the triples $t'_{i_2}, t'_{i_3}, ..., t'_{i_q}$ and include all the universe elements $a^z_{i_1}, ..., a^z_{i_q}$. Note that we can also decide to include only some of them – e.g. $(S^x_{i_1}, L^y_{i_1})$,$(S^x_{i_2}, L^y_{i_2}, a^z_{i_2})$, $(S^x_{i_3}, L^y_{i_3})$, ... – and this will let us handle the case in which some of the set selector gadgets included in M' share some elements.

Figure **??**fig:setselector shows an example of a simple set selector gadget.

We can force the constraint that at most k set selector gadgets can include their elements a^z_i adding exactly k new elements $E^x_1, ..., E^x_k$ to X and for each E^x_j add m triples that link them to the selector element of the m set selector gadgets:

$$\begin{array}{llllll} e_{1,1}: & (E^x_1, R^y_1, dum^z), & \ldots & e_{1,m}: & (E^x_1, R^y_m, dum^z) \\ e_{2,1}: & (E^x_2, R^y_2, dum^z), & \ldots & e_{2,m}: & (E^x_2, R^y_m, dum^z) \\ & \ldots & & & \\ e_{k,1}: & (E^x_k, R^y_1, dum^z), & \ldots & e_{k,m}: & (E^x_k, R^y_m, dum^z) \end{array}$$

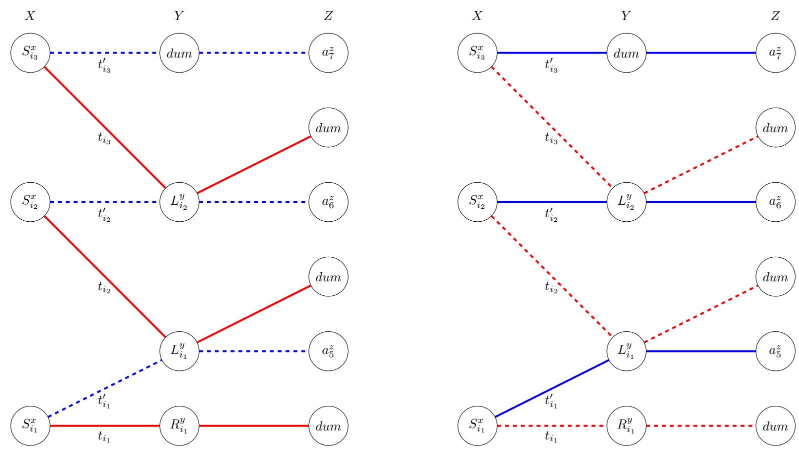

Figure 1: A simple set selector gadget for $S_i = \{a_5, a_6, a_7\}$, the triples are represented with edges of the same color. On the left the valid matching (solid red triples) when the triple $t_{i_1} = (S^x_{i_1}, R^y_{i_1}, dum)$ is choosen to include $S^x_{i_1}$; on the right the valid matching (solid blue triples) than can be built if the selector element $R^y_{i_1}$ is included elsewhere; note that this matching allows to include the universe elements $\{a^z_5, a^z_6, a^z_7\}$

Finally we can add as many triples $g_u = (G^x_u, dum^y, G^z_u)$ or (G^x_u, G^y_u, dum^z) as needed to *garbage collect* all the dum^y_u, or dum^z_u elements used in the construction above.

Figure 2 shows an example of the 3DMV construction for a simple unsolvable Set Cover instance.

($\Rightarrow$) Suppose that a valid solution M' exists for the 3DMV; then, in order to include all the elements E^x_j, exactly k of the triples e_{j_p} above (or one of the corresponding pairs) are included in M' . So at most k set selector gadgets are used to include all the elements $a^z_1, a^z_2, ..., a^z_n$ in M'. But, by construction, the corresponding sets S_i form a valid cover of A in the original SC problem.

($\Leftarrow$) In the opposite direction, suppose that a valid solution $S_{l_1} \cup S_{l_1} \cup ... \cup S_{l_k} = A$ of the SC problem exists. We build a vaild solution M' for the 3DMV problem picking the triples (E^x_i, R^y_i, dum^z) with $i \in \{l_1, ..., l_k\}$. This allows us to include all the universe elements associated with the set selector gadgets $S^x_i, i \in \{l_1, ..., l_k\}$, but the corresponding sets form a cover, so all elements a^z_j can be included. Furthermore as seen above if some sets have common elements, they can be included only in one of them. The remaining set collector gadgets corresponding to the sets S_r not included in the cover, can be included in the matching M' using the triples $t_{r_1}, t_{r_2}, ...$. Finally using the garbage collection triples we can include all the dum elements exactly once: for example, if the dum^y_u element is already included in the solution M' then we can pick the pair (G^x_u, G^z_u) instead of the full triple (G^x_u, dum^y_u, G^z_u).

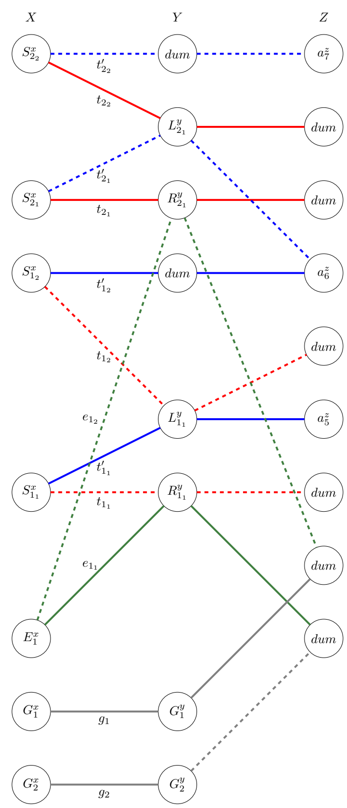

Figure 2: A simple example that shows the 3DMV construction corresponding to the unsolvable Set Cover instance: $k = 1, A = \{a_5, a_6, a_7\}, S_1 = \{a_5, a_6\}, S_2 = \{a_6, a_7\}$ (only two garbage collection triples are shown). Using the triples containing the element E^x_1 we can decide to include in the matching the universe elements $\{a^z_5, a^z_6\}$ or $\{a^z_6, a^z_7\}$, but not all of them.

3 Other variants

Note that if we add an integer k to the input and ask for a matching of size k or more ($|M'| \geq k$), like in the 3DM problem, and not for an exact cover; then the problem is solvable in polynomial time: indeed all the triples in M and the pairs in M_{XY}, M_{XZ} contain exactly an element of X, so the constraint is equivalent to having k or more elements of X in M'. But if M' is a valid matching of size k or more and we replace a triple (x, y, z) of M' with a pair (x, y) or (x, z) we get a valid matching M'' of the same size. So we can model our problem as a 2-dimensional matching problem with pairs from the set $A \subseteq X \times (Y \cup Z)$ (note that Y, Z are disjoint) built in the following way: $A = \{(x, y) \mid \exists (x, y, z) \in M\} \cup \{(x, z) \mid \exists (x, y, z) \in M\}$; which is solvable in polynomial time.

Also note that if we require X, Y, Z having the same cardinality: $|X| = |Y| = |Z|$ and ask for an exact cover then the problem is equivalent to the special case of 3DM in which $k = |X| = |Y| = |Z|$, which is again NP–complete [1]. With the cardinality constraint, all valid solutions M' must contain only triples: indeed M' must contain all the elements of X exactly once, so if it contains a pair, for example (x, y), then the number of elements of Z included in M' is $|\{(x, y, z) \in M'\}| + |\{(x, z) \in M'\}| < |Z|$, i.e. at least one member of Z remins excluded.

Finally the following problem, which adds a relaxed matching condition to the standard 2-dimensional matching problem, is also NP–complete:

Definition 3.1 (Relaxed 2-Dimensional Matching)**.**

Input: Set $T \subseteq X \times Y$ where X, Y are disjoint sets; given a triple $t_i = (x_j, y_p, y_q)$, we define $t_i^L = (x_j, y_p)$, $t_i^R = (x_j, y_q)$ and $T^L = \bigcup_i t_i^L, T^R = \bigcup_i t_i^R$.

Question: Does there exist a set $T' \subseteq T \cup T^L \cup T^R$ such that every element of $X \cup Y$ is included in a triple or a pair of T' exactly once?

The reduction from SC is analogous: we simply ignore the distinction between the sets Y and Z but still use distinct elements in the set Y of the Relaxed 2-Dimensional Matching problem to represent universe elements, selector elements and link elements.

References

[1] Garey, Johnson, and Tarjan. The planar hamiltonian circuit problem is NP-complete. *SICOMP: SIAM Journal on Computing*, 5, 1976.