Restriction of Exact Cover by 3-sets

Exact cover by 3-sets problem is defined as:

> **Instance**: a set $X = \{x_1, x_2, \ldots, x_{3n}\}$ and a family $F = \{(x_{i_1}, x_{i_2}, x_{i_3})\}$ of 3-elements subsets of X (triples);
> **Question**: Is there a subfamily F' of F such that every element in X is contained in exactly one triple of F'.

It is known that Exact cover by 3-sets problem is NP-complete even if input is restricted such that each element of X appears exactly in three triples.

> Is still NP-complete if the input is restricted further such that no pair of input triples share more than one element of X?

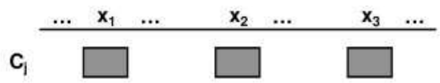
cc.complexity-theory

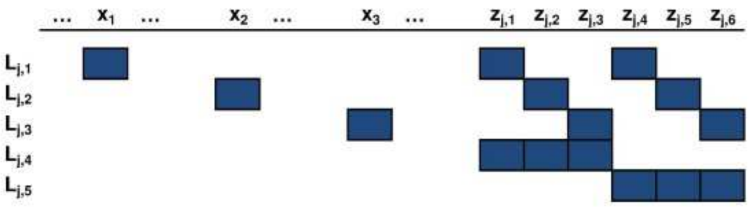
edited Dec 31 '13 at 12:04 asked Dec 31 '13 at 11:40
Mohammad Al-Turkistany
7,173 2 17 66

I didn't check if the Bangye's solution is correct/simpler, but a quick transformation from RESTRICTED X3C (the name is the same used by Gonzales) to **SINGLE OVERLAP RESTRICTED X3C** (the name is invented) that should work is:

- **Replace** each subset $C_j = \{x_1, x_2, x_3\}$ adding 6 new elements $z_{j,1}, z_{j,2}, \ldots, z_{j,6}$ and 5 new three elements subsets
  $L_{j,1} = \{x_1, z_{j,1}, z_{j,4}\}$,
  $L_{j,2} = \{x_2, z_{j,2}, z_{j,5}\}$,
  $L_{j,3} = \{x_3, z_{j,3}, z_{j,6}\}$,
  $L_{j,4} = \{z_{j,1}, z_{j,2}, z_{j,3}\}$,
  $L_{j,5} = \{z_{j,4}, z_{j,5}, z_{j,6}\}$.

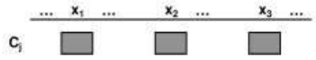
like in the figure below (blue triples). Informally, the three elements originally in C_j are grouped and in order to include elements $z_{j,1}, \ldots, z_{j,6}$ the exact cover must include the group of triples $L_{j,1}, L_{j,2}, L_{j,3}$ OR the two triples $L_{j,4}, L_{j,5}$, but not both.

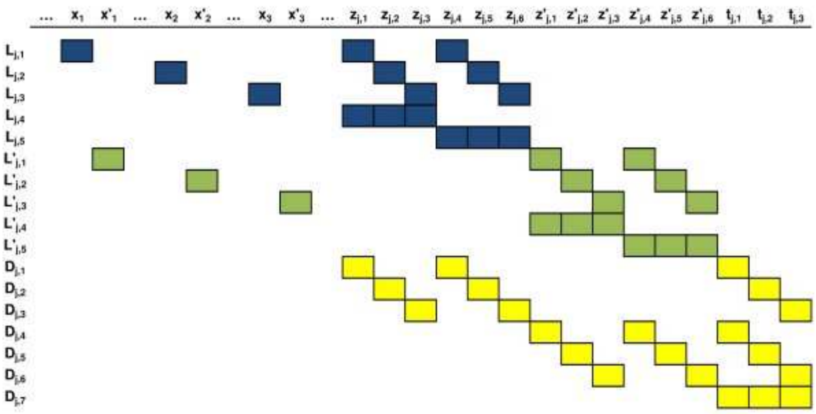




At this point no pair of triples share more than one element and each element is included in exactly 3 triples; except elements $z_{j,1}, \ldots, z_{j,6}$ which are included only in two triples.

- In order to fix this it's enough to add a duplicate of every element element $x_i \to x'_i, z_i \to z'_i$, a duplicate of each triple containing x_i or z_i elements using the corresponding duplicated elements (green triples in the figure below) and for each original triple C_j add three new elements $t_{j,1}, t_{j,2}, t_{j,3}$ and 7 new dummy subsets with elements:
  $D_{j,1} = \{z_{j,1}, z_{j,4}, t_{j,1}\}$,
  $D_{j,2} = \{z_{j,2}, z_{j,5}, t_{j,2}\}$,
  $D_{j,3} = \{z_{j,3}, z_{j,6}, t_{j,3}\}$,
  $D_{j,4} = \{z'_{j,1}, z'_{j,4}, t_{j,1}\}$,
  $D_{j,5} = \{z'_{j,2}, z'_{j,5}, t_{j,2}\}$,
  $D_{j,6} = \{z'_{j,3}, z'_{j,6}, t_{j,3}\}$,
  $D_{j,7} = \{t_{j,1}, t_{j,2}, t_{j,3}\}$ (yellow triples in the figure below).





(⇒) Suppose that $\bigcup_{j \in A \subseteq \{1,\dots,3n\}} C_j$ is an exact cover of the original RESTRICTED X3C instance. Then by construction:

$$\bigcup_{j \in A} (L_{j,1} \cup L_{j,2} \cup L_{j,3} \cup L'_{j,1} \cup L'_{j,2} \cup L'_{j,3} \cup D_{j,7}) \cup \bigcup_{j \notin A} (L_{j,4} \cup L_{j,5} \cup L'_{j,4} \cup L'_{j,5} \cup D_{j,7})$$

is an exact cover of SINGLE OVERLAP RESTRICTED X3C.

(⇐) Suppose that there exists an exact cover of the SINGLE OVERLAP RESTRICTED X3C instance. Every original element x_i must be included exactly once in the cover, but, as seen above, the only way to include an element x_i is by choosing a group of triples $L_{j,1}, L_{j,2}, L_{j,3}$ that correspond to an original triple C_j that contains x_i. Furthermore if $L_p, L_q, p \neq q$ are included in the exact cover we have $L_p \cap L_q = \emptyset$. So the collection $L_{j,k}$ of subsets in the SINGLE OVERLAP RESTRICTED X3C exact cover correspond to a valid cover $\bigcup C_j$ of the original RESTRICTED X3C instance.

The reduction can be done in polynomial time, so we can conclude that SINGLE OVERLAP RESTRICTED X3C is NP-complete.

Just note that a SINGLE OVERLAP RESTRICTED X3C instance built using the above reduction can contain two valid and *distinct* exact covers of the original RESTRICTED X3C problem, but we are sure that if only one exact cover exists, it can be "mirrored" to form a valid exact cover of SINGLE OVERLAP RESTRICTED X3C.

Let me know if you need a more formal proof for a paper.

edited Jan 3 at 7:36 answered Dec 31 '13 at 19:24

Marzio De Biasi
6,982 1 11 37

Thanks Marzio, I will give you my feedback on your reduction. – Mohammad Al-Turkistany Jan 1 at 13:26

Thanks Marzio for your nice reduction. Are you aware of other NP-completeness proofs that rely heavily on redundant encoding? – Mohammad Al-Turkistany Jan 6 at 11:25

Thanks :). For redundant encoding, do you mean something like the "duplicated elements trick" above? – Marzio De Biasi Jan 6 at 12:34

@MohammadAl-Turkistany: P.S. I'm going to post it on my blog, too; my English is not so good, do you think that "SINGLE OVERLAP RX3C" is a good name? Or perhaps it's better "single overlapping RX3C" or "single share RX3C",...? – Marzio De Biasi Jan 6 at 12:42

Yes. I mean "duplicated elements trick". Regarding the name, I suggest Unique overlap restricted X3C problem. – Mohammad Al-Turkistany Jan 6 at 19:05