Hidato is NP-complete

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Hidato (also known as Hidoku) is a logic puzzle game invented by Dr. Gyora Benedek, an Israeli mathematician. The rules are simple: given a grid with n cells some of which are already filled with a number between 1 and n (the first and the last number are circled), the player must completely fill the board with consecutive numbers that connect horizontally, vertically, or diagonally.

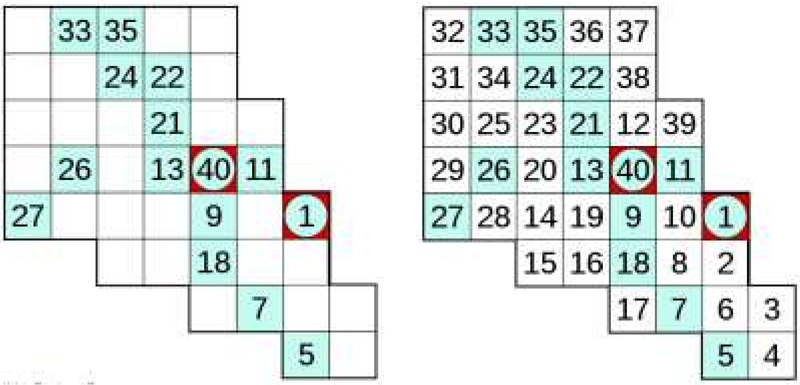


Figure 1: An Hidato game (that fits on a 8×8 grid) and its solution on the right.

We prove that the corresponding decision problem HIDATO : "*Given a Hidato game that fits in a* $m \times n$ *grid, does a valid solution exist?*" is NP-complete.

First we show that the problem is NP-hard using a polynomial time reduction from the Hamiltonian cycle problem on grid graphs (with holes) which is NP-complete [1]. Given a grid graph $G = (V, E)$, whose vertices are a subset of the square grid graph $Q = \{(x, y) : 0 \leq x, y \leq w\}$ (for simplicity we assume that the square has even sides $w = 2a$), we can identify the nodes of G with their integer grid coordinates: $V \subseteq \{(x, y),\ 0 \leq x, y < w\}$ (note that in Q there is always an extra column and an extra row) and without loss of generality we can assume that its upper left corner contains two nodes s, t at coordinates (0,0) and (0,1). If G has an Hamiltonian cycle then the edge (s, t) is part of it (red lines in Figure 2).

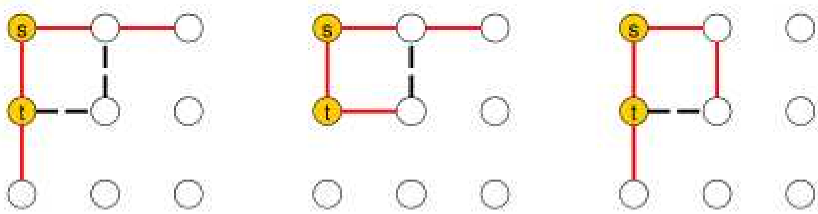


Figure 2: Possible traversals of the two top-left nodes of graph G.

- First we expand G adding the missing nodes of the embedding square: $H = \{(x, y) : (x, y) \in Q \setminus V\}$;
- then we double the coordinates of the points of both V and H and translate them by one: $(x', y') = (2x + 1, 2y + 1)$;
- we add a set K of $(w + 1) \times (w + 1)$ nodes with even coordinates: $K = \{(2x, 2y) : 0 \leq x, y \leq w\}$;
- finally we further expand K adding points $\{(x, y) : x, y \,\mathrm{odd}\}$ and $\{(x, y) : x, y \,\mathrm{even}\}$ to make $V \cup H \cup K$ a *diamond*.

The construction steps are shown in Figure 3.

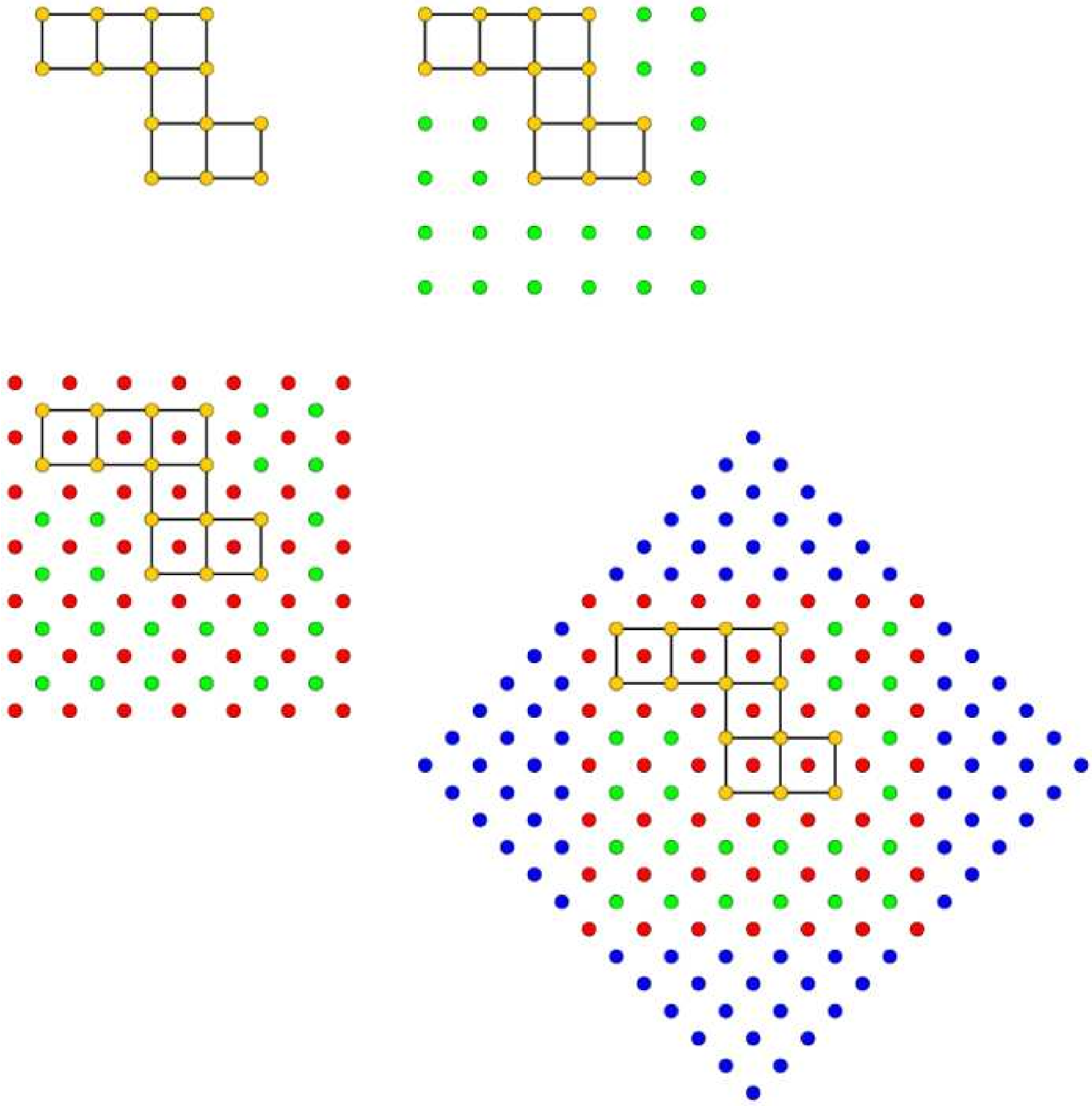


Figure 3: Yellow nodes represent the original graph G, green nodes represent the nodes in H, red nodes represent the nodes in K, blue nodes represent the final expansion to make the whole graph a "*diamond*" .

Now starting from point $P_1 = (0, 0)$ we can build a path P adding $|H| + |K| - 1$ edges: we first visit $P_2 = (1, -1)$ and $P_3 = (2, -2)$ then we join vertically the points

$P_i = (x, y) \rightarrow P_{i+1} = (x, y+2)$ in K (downward direction), taking care to inlude points in H when they are present on the left: $(x, y) \rightarrow (x-1, y+1) \rightarrow (x, y+2)$. Once we reach the bottom of the diamond we can leave an unconnected node and continue joining points $(x, y) \rightarrow (x, y-2)$ (upward direction). Once reached the right part of the diamond we can "transfer" to the left using the unconnected nodes at the bottom and connect in a similar manner the leftmost part until we reach point $P_c = (0, 1),\ c = |H| + |K|$.

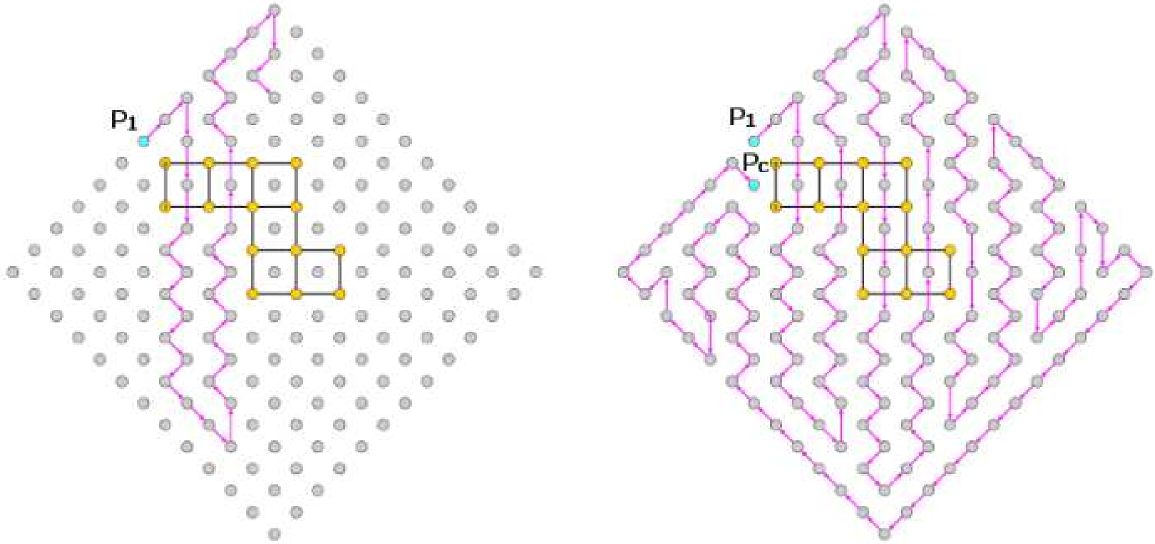
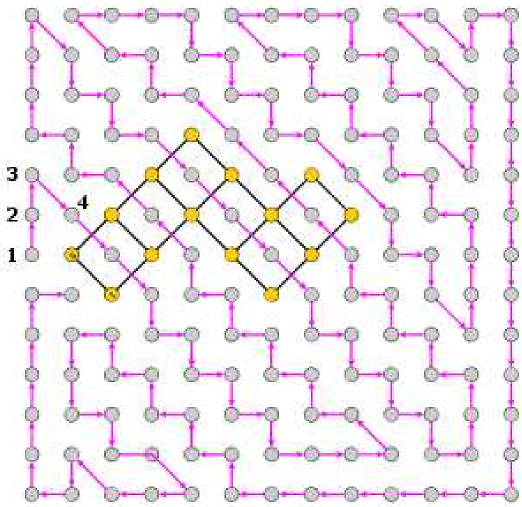


Figure 4: The construction of path $P_1, P_2, \ldots, P_c$.

If we rotate the diamond 90 degrees counterclockwise, we get a $(2w+1) \times (2w+1)$ square grid Q' where *each cell corresponds to a node*. We can mark the points P_i of the path with numbers $1, 2, \ldots, c$, and the node s of the original graph G (at coordinates $(1, 1)$) with number $c + |V| = (2w+1) \times (2w+1)$. The other $|V| - 1$ nodes are left empty. Q' is a Hidato game and it has a solution if and only if the graph G has an Hamiltonian cycle.

($\Rightarrow$) suppose that Q' has a valid solution; then the first number of the solution is the node t of the original graph and it must be numbered $c+1$ (previous numbers are all fixed); then the next empty cells can be reached only through the diagonals of Q' which correspond to the orthogonal edges of G. Each cell is uniquely numbered, and the final cell of the solution is s (numbered with $c + |V|$), so the sequence of cells determines a simple path from t to s in G, but the two nodes are connected, so it is also an Hamiltonian cycle.

($\Leftarrow$) suppose that G has an Hamiltonian cycle, then it is enough to follow it from node t to node s and mark the corresponding cells with number $c+1, c+2, \ldots, c+|V|$. By construction the numbers $1, 2, \ldots, c$ are valid, and the next numbers (corresponding to the Hamiltonian cycle) are valid, too, because orthogonals moves on G correspond to valid diagonal moves on Q' .



33	66	67	68	69	94	95	96	97	107	108	110	111
32	34	65	64	70	71	93	92	98	99	106	109	112
31	35	36	63	62	72	73	91	90	100	101	105	113
30	29	37	38		61	74	75	89	88	102	104	114
3	28	27		39		60		76	87	86	103	115
2	4		26		40		59		77	85	84	116
1	169	5		25		41		58	78	79	83	117
154	155		6	24	23		42	57	56	80	82	118
153	138	137	7	8	22	21	43	44	55	54	81	119
152	139	136	135	9	10	20	19	45	46	53	52	120
151	140	141	134	133	11	12	18	17	47	48	51	121
150	147	142	143	132	131	13	14	15	16	49	50	122
149	148	146	145	144	130	129	128	127	126	125	124	123

Figure 5: The final Hidato game Q' which has a solution if and only if the starting graph G has an Hamiltonian cycle.

The construction of Q' from G can be done in polynomial time (the sets H, K can be calculated in $O(w^2)$, and the calculation of the initial fixed c numbers takes $O(w^2)$ time, too, so Hidato is NP-hard.
If we assume that the input of the game is given with an array of $m \times n$ integers containing the initial fixed numbers of the cells (0 if the cell is empty), then it is easy to see that a solution can be verified in polynomial time ($O(m \times n)$).

So if the input is given as an array of integers and we drop the unique solution constraint Hidato is NP-complete.

□

A final note: if we add the constraint that the solution must be unique then the problem is US-hard (Unique Polynomial Time), but US contains co-NP, so it is co-NP-hard, too and therefore unlikely to be in NP.

[1] Alon Itai, Christos H. Papadimitriou, Jayme Luiz Szwarcfiter: Hamilton Paths in Grid Graphs. SIAM J. Comput. 11(4): 676-686 (1982)

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